CONDUCTION AND RADIATION

K. Muralidhar
Department of Mechanical Engineering
Indian Institute of Technology Kanpur
Kanpur India

J. Banerjee
Department of Mechanical Engineering
Sardar Vallabhbhai National Institute of Technology
Surat India

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Overview

Thermodynamics adopts the following viewpoint: Heat is energy in transit and is transported by virtue of a temperature difference. The subject of heat transfer is an extension and is concerned with the calculation of the *rate* of heat transfer for a *given* temperature difference. This step brings into picture the physical mechanisms in the intervening material (or, space) when it is subjected to a temperature difference. If the medium is a solid body, the dominant transport mechanism is *conduction*; conduction is also the prevelant mechanism in a stagnant fluid. In an engineering context, fluids are often in a state of motion, driven by an external mechanism or by internal density differences. An additional (bulk) transport mechanism (called *advection*) is thus introduced over and above conduction. Jointly, the heat transfer mechanism in fluids comprising conduction and advection, is referred to as *convection.* If temperature difference is applied across surfaces that enclose evacuated space, heat transfer occurs *radiation*, namely by exchange of energy between surfaces in the form of electromagnetic radiation.

In the mathematical description of heat transfer, it is necessary to assume that the transported quantities such as temperature and heat flux, as well as material properties vary continuously from one point to another. In view of the particulate nature of matter, it is appropriate to question the validity of this assumption. Equations derived on the basis of this assumption have, however, withstood the test of time (mainly via experiments) and the treatment of the physical medium as a continuum has firmly become established. The continuum approximation breaks down if the density of the medium is so small that molecular motion occurs on the same scale as the total distance between the energy exchanging surfaces. If λ_g is the mean free path of the molecules and L the characteristic dimension of the macroscopic system, the continuum approximation is valid if $\lambda_g/L \ll 1$ and not valid if $\lambda_g/L \sim 1$, order of unity (or greater). In much of the discussion to follow, physical phenomena are assumed to take place at the *continuum* scale.

The ratio λ_g/L is called *Knudsen number.* It is of order unity in rarefied gases, for example in the upper layers of the earth's atmosphere. The formulation of the heat transfer problem at this limit of the Knudsen number is not covered here. A short discussion on microscale heat conduction is however, presented.

Typical atomic length scales are of the order of 10^{-9} m, time scales are smaller than 10^{-9} seconds, and frequencies are in excess of 10^9 Hz. In view of the extreme values of these scales, the continuum approximation has wide applicability. Consider the following example. Under atmospheric conditions near the surface of the earth, 1 mm^3 volume of air contains around 3×10^{16} molecules. This is a very large number. Over the length scales involved in engineering problems, 1 mm^3 is a small volume and yet it is sufficiently populated for the continuum approximation to be valid.

The continuum scale is also called the *macro-scale.* The commonly observed laws of physics such as conservation of mass, Newton's second law of motion, and conservation of energy are applicable here[1]. Length and time scales that are smaller are called the *micro-scales.* While the principle of conservation of energy is applicable at the micro-scale, one can expect a significant departure in terms of the material behaviour. When the *atomistic* scale is reached, departures are to be expected in terms of material behaviour as well as the physical laws.

In a problem involving continuum scale heat transfer, mass and energy are unrelated quantities. Hence, the conservation of mass principle is not required in the development of the mathematical model. In conduction and radiation heat transfer, the velocity vector is mostly zero. Hence, the momentum equation is also not required during the theoretical development.

The three modes of heat transfer, conduction, convection and radiation can be jointly categorized in terms of a *mean free path.* It is a representative distance over which elementary particles travel before transporting thermal energy to their neighbourhood. For conduction, it is a small value; for radiation it is large. In solids, the mean free path can be described by theories analogous to kinetic theory of gases. In contrast, thermal radiation transports as electromagnetic waves below the infrared range of wavelengths. The mean free path takes on intermediate values for convection and is determined by the nature of fluid flow. It is also to be emphasized that all three mechanisms can be jointly present in certain applications.

Principles of thermodynamics necessarily must appear in heat transfer analysis. The first law is enforced in the form of a conservation of energy principle. In applications presently considered, the work contribution is generally negligible, or identically zero. The second law prevails in a more subtle manner. It dictates the direction of energy transfer in relationship to the temperature gradient. In other contexts, it can require material properties to satisfy certain constraints.

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[1] These laws also hold at large scales though there is ambiguity at truly astronomical length and time scales.

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Part I: CONDUCTION

Chapter 1

Energy Equation

The first law of thermodynamics, equivalently the principle of conservation of energy can be expressed in analytical form by first utilizing the mathematical identity called the *Reynolds Transport Theorem.*

1.1 Reynolds transport theorem

This theorem relates integrals of transport properties evaluated over a definite mass of material occupying an arbitrary volume with integrals over fixed well-defined volumes. It is a useful mathematical identity because physical laws are stated for fixed mass systems, while it is easy to identify fixed control volumes in engineering applications.

Let Φ be a transported quantity associated with the physical material occupying a space SP and ϕ be its intensive value, that is, Φ per unit mass. Since Φ is defined with respect to a fixed mass of the material, conservation principles such as the first law of thermodynamics are applicable to it. Let CV be a control volume fixed in space and CS be the surface (control surface) enclosing it. The control volume is chosen to overlap the material space SP at a time t. The quantities Φ and ϕ are related as

$$\Phi = \int_{SP(t)} \rho\, \phi\, dV = \int_{CV} \rho\, \phi\, dV$$

where ρ is the material density. Let $\mathbf{u}$ be the velocity vector and $\mathbf{n}$, a unit outward drawn normal on CS (Figure 1.1). The Reynolds transport theorem (RTT) can be stated in this context as

$$\frac{d\Phi}{dt} = \int_{CV} \frac{\partial}{\partial t} (\rho\, \phi)\, dV \; + \; \int_{CS} \rho\, \phi\, \mathbf{u} \cdot \mathbf{n}\, dS \tag{1.1}$$

i.e., the rate of change of the total property Φ $(= d\Phi/dt)$ of a particular mass of material enclosed by the space SP is the sum of contributions from changes in ϕ throughout the control volume and the net outflow of ϕ at the control surface.

Chapter 2

Steady Heat Conduction in One Dimension

In the present chapter, we review analytical solutions of the steady heat conduction equation in various coordinate systems.

2.1 Slab and circular geometries

In many instances, the geometry of the physical domain is such that temperature changes rapidly with respect to one coordinate but is relatively constant in the other directions. Since the corresponding fluxes (for nearly uniform temperature) would be small, energy transfer is predominantly in the direction along which large changes in temperature take place. It is then sufficient to work with the heat conduction equation that neglects temperature gradients in some directions but retains them in others.

Consider first a *slab* geometry (Figure 2.1) where temperature changes take place primarily in the x-direction. Assuming the slab material to be isotropic and homogeneous, the steady three dimensional heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{2.1}$$

reduces to the one dimensional form

$$\frac{d^2 T}{dx^2} = 0 \tag{2.2}$$

For the boundary conditions shown, we have

$$T(x = 0) = T_1 \qquad T(x = L) = T_2$$

Chapter 3

Steady Heat Conduction in Two Dimensions

Temperature distribution in a two dimensional homogeneous, isotropic region can be obtained by solving the heat conduction equation in an appropriate coordinate system. Cartesian and cylindrical coordinate systems are of interest to the present chapter. The analytical technique is broadly based on the method of *separation of variables.* If the physical region is inhomogeneous, the thermal conductivity (and specific heat) would be functions of position. The methods discussed below will then, not be directly applicable. However, for certain configurations where the partitioning is conveniently along constant coordinate lines, an analytical solution should be possible. If thermal conductivity is a function of temperature (but not position), the governing equation can be linearized using Kirchoff's transformation (Section 2.1). The resulting equation is once again analytically solvable. If the region has a thermal conductivity that depends on temperature as well as position, the equation can only be numerically solved.

3.1 Heat conduction in a square domain

Consider determination of temperature distribution in a square region shown in Figure 3.1(a). The mathematical problem can be stated in Cartesian coordinates as follows:

$$T_{xx} + T_{yy} = 0 \tag{3.1}$$

where suffix indicates partial differentiation. In Figure 3.1(a), the boundary conditions for temperature are

$$T(0, y) = T(x, 0) = T(x, 1) = 0; \quad T(1, y) = 1$$

Here, temperature is represented in dimensionless form (with maximum corresponding to unity and minimum being zero). The edge of the square region is used as the length scale. The above mathematical problem can be solved by separation of variables, a

Chapter 4

Unsteady Heat Conduction in 1 and 2D Regions

In the previous chapter, steady heat conduction in one and two dimensional regions was analyzed. The analytical tool used was essentially separation of variables, leading to an exact solution in the form of an infinite series. This approach is applicable to materials that have a constant thermal conductivity. Of course, it is restricted to simple geometries such as rectangles, circles and spheres. Extension to composite regions in which thermal conductivity is piecewise constant is possible by applying flux and temperature continuity conditions at the interfaces. For materials that show a temperature-dependent thermal condictivity, the steady heat conduction equation can be linearized using Kirchoff transformation.

The separation of variables approach continues to hold for unsteady heat conduction in orthogonal coordinate systems when the material has constant thermal conductivity. Nonlinear problems require special treatment and the solutions are at best approximate. For linear heat conduction, the eigenvalue problem will emerge purely in the spatial direction since the time axis stretches to infinity. These issues are discussed through specific examples in the following sections.

4.1 Bounded one dimensional domains

Consider heat conduction in a slab that is initially at a temperature T_2; the left face temperature is suddenly raised to a temperature T_1 and a thermal front diffuses through the slab as a function of time. The steady state temperature distribution is linear, varying from T_1 on the left face to T_2 on the right face. The distribution is not linear during the transient evolution process.

In a one dimensional region such as a constant conductivity slab (Figure 4.1), the unsteady heat conduction equation along with initial and boundary conditions is stated

Chapter 5

Phase Transformation

For pure metals as well as eutectic alloy compositions, the phase change boundary is a sharp surface[1]. On one side of the surface, the material is in liquid form, while it is a solid on the other. For alloys, phase change occurs over a region of finite thickness[2]. The analysis of phase change in alloys is not the subject of discussion.

5.1 One dimensional phase change

The present chapter addresses heat transfer with phase change in one dimensional geometries. Here, the solid-liquid interface is planar and moves strictly in one direction. Thus, the phase boundary overlaps a constant coordinate line. This requirement is essential for the derivation of analytical solutions. In two and three dimensions, the phase boundary is curvilinear, in general and applying boundary conditions can be a source of difficulty. Thus, analytical methods discussed in the context of one dimensional problems do not carry over readily to two and three dimensions. Real-life phase change problems are invariably handled *via* numerical simulation by a suitable discretization procedure.

At a first glance, change of phase of the material can be thought of as a situation analogous to contact resistance between dissimilar materials. There are important differences, however. These are:

1. Temperature at the phase change front is the fusion point temperature of the material and is pre-determined. Additionally, the temperature field is continuous as one proceeds from the liquid phase to the solid. This trend is in contrast to a contact resistance wherein a jump in temperature is possible.

2. Unlike temperature, heat flux is not continuous since latent heat is released or absorbed at the solid-liquid interface. The change in heat flux across the interface

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[1] of zero thickness

[2] called the *mushy zone*

Chapter 6

Diffusive Mass Transfer

Consider fresh water floating over a concentrated salt solution at an initial instant of time as in Figure 6.1 With the passage of time, salt diffuses from the base of the beaker into the rest of the liquid. In a modeling framework, the physical property that is transported is salt concentration, with water serving as a solvent. The transport process is in the direction of diminishing concentration. In this respect, concentration is a material property analogous to temperature. In fact, once can conceptualize mass diffusion (of salt) as a *gradient diffusion process* quantified by the mass transfer rate

$$\dot{m}'' = -\rho D \nabla C \tag{6.1}$$

where C is salt concentration (kg of salt/kg of solvent), ρ is the density of the solvent[1] (kg/m^3), D is the diffusivity of salt in water (m^2/s) and $\dot{m}''$ is the mass flux in units of kg/m^2-s. The above expression, called *Fick's law* is analogous to Fourier's law of heat conduction. Mass diffusivity is a property of the solute as well as the solvent. The product $\rho \times D$ plays a role akin to thermal conductivity in a heat transfer process.

An *analogy* refers to the a situation wherein two distinct physical phenomena give rise to identical mathematical models. One often encounters analogies in fluid and thermal sciences. For example, transport of mass, momentum and thermal energy are quite similar under certain conditions. Of interest to the present discussion is the mathematical analogy between diffusion of thermal energy and diffusion of mass in a physical domain. When an analogy holds, analytical techniques applicable to one physical problem can be fruitfully used to solve the other. Such analogies across dissimilar disciplines is also common-place. Examples are electromagnetic field theory, semiconductor physics, and transport phenomena in porous media. Once the governing equations are setup, methods of solution of one discipline carry over to the other.

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[1] water, in the present example

Chapter 7

Inverse Heat Transfer

Engineers are often asked to estimate the performance of a given device, either *via* simulation or through experiments. The implication here is that one would like to gage those quantities that constitute the device output for a given set of specified (input) parameters. This information would be called the *performance characteristics* of the device. An example is efficiency as a function of head, for a pump or a turbine, at specified flow rates and RPM. This is referred to as the *direct* or the *forward* problem. In an independent scenario, the engineer is asked to build a system wherein the performance characteristics are pre-determined. This is the *inverse* problem; clearly it is the central theme of the design process.

The direct and the inverse problems are inter-related since they refer to the same application, but differ vastly in the manner in which they can be addressed. For example, it should be clear that the inverse problem need not have a *unique* solution. Several configurations can be arranged to generate a given set of output characteristics. To generate specific results, it is necessary to impose additional *constraints* such as size, weight, and cost to force a unique solution. In an extraordinary situation (for example, when the number of constraints is large) no configuration may be acceptable at all; here the solution simply does not *exist.* These factors make the treatment of the inverse problem quite complex. In the present chapter, some examples of the inverse formulation in an analytical framework are presented. The method of solution has been discussed. The material is presented from a thermal science perspective, though tools referred to have a broader applicability[1].

7.1 Parameter estimation

In words, the direct problem can be posed in the following manner: Given complete information regarding the input data, in general a function of time, the process parameters and the material properties, find the output response of the system (pointwise

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[1] The extension of the analytical techniques of the present chapter to the design of full engineering system is a research topic under consideration.

Chapter 8

Microscale Heat Conduction

The term *micro* in the chapter title refers essentially to the smallness of the length scale, though it is connected to small time scales as well. In metals and alloys, the distinction between small and large length scales is with reference to lattice dimensions. For example, thin films used in electronic devices or as optical coatings may be a few microns thick (or smaller). Small linear dimensions are encountered when the overall heat transfer in the physical domain occurs for a very short duration or occurs repeatedly at a high frequency[1]. Small time scales encountered in applications go from microseconds (10^{-6} s), nanoseconds (10^{-9} s), picoseconds (10^{-12} s) to femtoseconds (10^{-15} s). High frequencies would be megahertz (10^{6}), gigahertz (10^{9}) and terahertz (10^{12}). These frequencies should be compared with those encountered in traditional applications, say IC engines that run at 2400 rpm and gas turbines that run at (say) 24,000 rpm. In units of Hz, these values are 40 and 400 respectively; classical heat conduction is entirely satisfactory in this context.

The relationship between distance and time (and hence, frequency) can become clear from estimates of the thermal boundary-layer thickness. For a semi-infinite solid, it was shown that

$$\delta \approx \sqrt{t} \quad \text{(step response)}$$

and

$$\delta \approx \frac{1}{\sqrt{\omega}} \quad \text{(oscillatory response)}$$

The proportionality constants in the above expressions involve the transport properties of the material. These expressions show that the size of the thermally affected region increases as the square root of time in a transient process and as the reciprocal of the square root of the forcing frequency in a pulsatile heating problem. In addition, small time scale, small length scale and high frequency correlate with one another.

At the length and time scales referred above, it can be shown that the principle of *conservation of energy* continues to be applicable as for the continuum limit. However, it is quite important that the structure of the material is accounted for.

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[1] a process referred to as *ultrafast*

Chapter 9

Introduction to Radiative Transfer

One way of distinguishing radiative energy transfer from heat conduction is as follows. We associate a long mean free path to the radiant energy carriers, as opposed to a short mean free path for heat conduction. The energy carriers in radiation are photons; in heat conduction, they are normally electrons moving in a phonon environment. In semiconductors, phonons (lattice vibrations) may serve as energy carriers as well. A large mean free path has the following implication. The energy carriers travel long distances before experiencing a change, either due to interaction with other photons, with the intervening medium, or a surface. The change referred here could be that in wavelength, energy density or direction. A simple example where the mean free path is clearly large is energy transfer between surfaces held in vacuum. Here, a photon leaving a surface travels unhindered till it strikes the second. The heat exchange calculations in this context must refer to surface-to-surface exchanges, temperature profiles across surfaces being less meaningful in this context. A similar situation is encountered during energy transfer in media transparent to the wavelength under question. Of course, there are examples such as dense particle-laden fluids, cloudy atmosphere and ionized gases where radiation interacts with the surrounding fluid during its passage from one surface to the other. The medium is then said to *participate* in the energy transfer process. It is also conceivable that, under extreme conditions, all radiation emerges from the body of the fluid and none from the bounding surfaces. A temperature profile will then appear in the physical domain, as in the context of heat conduction.

The initial discussion on radiative transfer (Chapters 9-15) assumes that the intervening space is vacuum or is purely transparent to energy exchange. Thus, the analysis is restricted to surface-to-surface heat transfer. Chapter 16 onwards, radiative transport in a participating medium is described.

Consider heat transfer from and to an opaque solid object first. The physical mechanism responsible for energy release from a surface is electronic transitions within the material. In the interior, the emitted radiation is internally absorbed or scattered and is accounted for in the conduction mechanisms. If the surface is exposed to vacuum

Chapter 10

Black Body Radiation

A true black body is an idealization but a close representation is possible with carefully constructed laboratory models. The advantage of using the idea of a black body is that it provides a standard against which real surfaces can be compared. This comparison is possible because emission from the black body also permits a first principles derivation using quantum mechanics[1]. In the present chapter, we utilize results of such an analysis, but the derivation is beyond the scope of the book To this extent, the emission law from a black body is stated here as a *constitutive relationship* for radiative heat transfer. Engineering problems in radiation can then be solved by combining the constitutive relation with energy balance laws.

Prevost's law states that energy is emitted by all surfaces whose temperatures exceed zero kelvin. The *emissive power* of a surface is calculated on the basis of the energy emitted per unit area of the surface and unit time. Emissive power is expressed in units of W/m^2. As in the case of the surface properties, it is possible to define a spectral directional emissive power to be the radiation emitted per unit time and unit area in the wavelength interval λ to $\lambda + d\lambda$ and angle $d\omega$ centered around the direction ω. The spectral hemispherical emissive power is the spectral directional emissive power summed over all angles, equivalently integrated over all directions[2]. The total hemispherical emissive power is the spectral directional emissive power integrated over all ω and all wavelengths $0 < \lambda < \infty$. The present chapter deals with the emissive power of a black body. Properties of real surfaces are discussed in Chapter 11. The suffix b is used to denote properties of a black surface. Hence, for a black body, the following notation is made use of:

1. $e'_{\lambda b}$ is the spectral directional emissive power (W/m^2-μm-sr).
2. $e_{\lambda b}$ is the spectral hemispherical emissive power (W/m^2-μm).
3. e'_b is the total directional emissive power (W/m^2-sr).

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[1] The analysis is based on the work of Max Planck, a German physicist.

[2] The direction ω referred here is in spherical coordinates defined in terms of θ and ϕ. This point is discussed in Chapter 9.

Chapter 11

Properties of Non-black Surfaces

A black surface has spectral emission characteristics that follow Planck's law. A non-black surface, by definition has emitted radiation whose spectrum does not follow the black body spectrum. It can then be shown that the emissions of non-black surfaces are always less than that of a black body (for each wavelength and direction)[1].

Remarks:

1. All real life surfaces are non-black. Some of them approach black body behaviour under certain limiting conditions[2].
2. Since emissions from real surfaces are wavelength and direction dependent, the surface properties are also functions of wavelength and direction. Under exceptional circumstances, this functionality can be ignored and one can work with average values.
3. For a black surface, $\alpha = \epsilon = 1$ and $\rho = \tau = 0$. This statement holds for every wavelength and every direction. Hence, for a black surface

$$\epsilon'_{\lambda b} = \epsilon_{\lambda b} = \epsilon_b = 1$$

$$\alpha'_{\lambda b} = \alpha_{\lambda b} = \alpha_b = 1$$

$$\rho'_{\lambda b} = \rho_{\lambda b} = \rho_b = 0$$

$$\tau'_{\lambda b} = \tau_{\lambda b} = \tau_b = 0$$

For a real surface (each wavelength, each direction), $\alpha < 1, \epsilon < 1$, $\rho > 0$ and $\tau > 0$. In addition, real surface properties can also depend on temperature. In practice, surfaces degrade with age and hence the radiation properties are time-dependent as well.

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[1] This remark is related to the statement of Chapter 9 that a black body is the greatest absorber of radiation in every direction and at every wavelength.

[2] Examples are surfaces with coatings and roughened surfaces.

Chapter 12

Energy Exchange between Black Surfaces

Radiative exchange of thermal energy between surfaces that are at distinct temperatures will be of interest to the present as well as succeeding chapters. For the present discussion, the surfaces are taken to be black. The surface properties do not appear in the formulation since $\alpha = \epsilon = 1$ while $\rho = \tau = 0$. The intervening medium is taken to be vacuum (or gas) for which the refractive index is (practically) unity. Thus, energy exchanges accounting for all wavelengths and all directions (*hemispherical total exchanges*) are considered[1]. For a black surface, knowledge of the total emissive power e_b is equivalent to knowing temperature via the Stefan-Boltzmann relation. With T given, the spectral emissive power $e_{b\lambda}(\lambda, T)$ can be recovered from Planck's black body function. Hence, for black surfaces, spectral and total exchange calculations are equivalent.

Figure 12.1 shows two surfaces in the vicinity of each other that exchange thermal energy by radiation. These surfaces are labeled 1 and 2 in the figure. Consider point P_1 on surface 1 where the temperature is T_1. An elemental area dA_1 is identified on surface 1 with point P_1 at its center. The corresponding location on surface 2 is P_2 with an elemental area dA_2 and a temperature T_2. The line joining points P_1 and P_2 is $\overline{P_1P_2}$, a distance indicated by the symbol r. The angle made by $\overline{P_1P_2}$ with the unit normal on surface 1 is θ_1; the corresponding angle on surface 2 is θ_2. These angles are measured in the clockwise direction.

Proceeding with the energy exchange calculations, the expression for energy emitted can be derived as follows (also see Problem 11).: For the element dA_1 in the direction $\overline{P_1P_2}$ per unit solid angle per unit time, the emission is $e'_{b1}dA_1{=}i_{b1}dA_1\cos\theta_1$. This expression is written in terms of radiation intensity because of the property of isotropy of black body radiation. Dependence of emissive power on wavelength is presently ignored (see Chapter 15 for a spectral formulation). The quantity $dA_1\cos\theta_1$ is the elemental area projected normal to the direction of $\overline{P_1P_2}$. In addition, we can use the result $i_b = e_b/\pi = \sigma T^4/\pi$ of Chapter 10.

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[1] Consequently, the emissive power of a surface at a temperature T is σT^4.

Chapter 13

Energy Exchange between Diffuse-Gray Surfaces

Diffuse-gray surfaces have absorptivity and emissivity that are constants with respect to wavelength and direction. The constant values are, of course, less than unity, the maximum permissible value. In addition, Kirchoff's law $\alpha = \epsilon$ continues to be applicable (see discussion in Section 11.2.1). In the present chapter, we continue to look at examples where the transmittivity of the surface is zero. Thus, the incident radiation is partly absorbed while the rest is reflected. The reflectivity of the surface is simply $\rho = 1 - \alpha = 1 - \epsilon$. The reflected radiation is taken to be *diffuse*[1], namely distributed uniformly in all directions. In this respect, the emitted radiation is indistinguishable from the reflected. For a diffuse surface, a given surface will view all other surfaces without preference, to the extent permitted by the shape factor.

Real surfaces can only be approximated as diffuse-gray since some dependence on wavelength and specularity (non-diffusive behaviour) is invariably obtained. In addition, real-surface properties can be functions of temperature itself.

Apart from a non-zero reflectivity, a second point of departure of a gray surface from black is the amount of radiation leaving the surface. The emitted radiation is the product of emissivity and the emission of a black body at the surface temperature, namely $\epsilon \times e_b$. The reflected part is the product of reflectivity and the radiation incident on the surface, namely $\rho \times G$. Here, the symbol G refers to the incident radiation (Figure 13.1) and is called *irradiation.* The total radiation leaving the surface, called *radiosity* is given a symbol J and expressed as

$$J = \epsilon e_b + \rho G \tag{13.1}$$

Both G and J are heat fluxes with units of W/m^2. An energy balance applied to the surface shown in Figure 13.1 yields

$$\frac{Q}{A} = J - G$$

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[1] as opposed to *specular*

Chapter 14

Treatment of Variable Wall Temperature

When some or all surfaces of an enclosure are prescribed with a temperature distribution (instead of a constant temperature), it is convenient to work with elemental areas and the energy exchanges among them. Each elemental area at a location x can be taken to be at a uniform temperature $T(x)$. Over this elemental area, the local heat flux $q(x)$ is also uniform. The energy exchange calculation for the enclosure is then obtained by integrating local heat fluxes over the area of the entire edge. As an example, let the k^{th} surface of the enclosure be given a temperature distribution $T_k(x_k)$, where x_k is a local coordinate over the k^{th} surface. The local heat flux is denoted as $q_k(x_k)$. Let us also assume that the enclosure surfaces are all flat and black. An equation of the type (Equation 12.7)

$$Q_k = \sigma T_k^4 A_k - \sum_{j=1;j\neq k}^{j=N} \sigma T_j^4 A_k F_{kj}$$

is now re-written as

$$q_k(x_k)dx_k = \sigma T_k^4(x_k)dx_k - \int_s \sigma T^4(x_s)dx_k dF_{dk-ds}$$

Here, dx_k is an element on surface k where the energy balance calculations are being conducted. The coordinate s is tangential to the enclosure surface everywhere. The interpretation of this equation continues to be: external heat addition = energy leaving the surface − energy incident on it. The above representation is exact since the energy balance expression is now applied to a differential element rather than a full edge of an enclosure.

It is convenient to divide the enclosure into N surfaces where temperatures could be spatially constant, as with the ambient, or have surfaces of known radiation properties.

Chapter 15

Treatment of Non-Gray Surfaces

Non-gray surfaces have properties such as absorptivity and emissivity that are dependent on the wavelength of radiation. The radiosity and heat flux formulations of Chapters 13 and 14 carry over with certain modifications[1]. These aspects are discussed in the present chapter.

15.1 Spectral radiosity formulation

When surface properties exhibit strong dependence on wavelength, the following changes need to be incorporated into the governing equations.

1. The surface properties are wavelength dependent, i.e. we use functions such as $\epsilon_\lambda(\lambda)$ in the radiative energy balance equation.
2. The wavelength-dependent radiosity J_λ is defined as follows: $J_\lambda d\lambda$ stands for the net radiation leaving a surface in the wavelength interval λ and $\lambda + d\lambda$. J_λ has units of W/m^2-μm. The external heat transfer is analogously represented in a spectral calculation as $Q_\lambda d\lambda$ where Q_λ has units of W/μm and the spectral heat flux q_λ has the units of spectral radiosity.
3. The black body emissive power $e_b = \sigma T^4$ (in units of W/m^2) is now replaced by the spectral hemispherical emissive power $e_{\lambda b}$ (in units of W/m^2-μm). It is given by the Planck's formula of Chapter 10, namely

$$e_{\lambda b}(\lambda, T) = \frac{2\pi C_1}{\lambda^5 \left(\exp(C_2/\lambda T) - 1\right)}$$

4. Calculations for the k^{th} surface of an enclosure repeated over each wavelength can then be combined to determine the total external heat exchange (or total

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[1] The problem statement of heat exchange in terms of wavelength is often called *spectral*.

Chapter 16

Gas Radiation

Several gases and liquids are practically transparent to visible radiation. The transmittivity is close to unity, the associated absorptivity being close to zero. In the presence of suspended particles, light is scattered and transmittivity is no longer unity. Scattering is generally uniform in all directions[1]; it redirects energy from a given direction to others. Consequently, energy transmitted in a particular direction is reduced. One can then think of scattering as equivalent to absorption of energy.

Even in the absence of particles, molecules of a gas can scatter radiation at specific wavelengths. Thus, when thermal radiation is exchanged, it is possible that the gaseous medium shows a non-zero absorptivity for certain wavelengths, being purely transparent to others.

The process of absorption of energy in a gaseous medium is well-modeled by the expression (Figure 16.1)

$$\frac{I_\lambda}{I_{\lambda,0}} = \exp\left(-\kappa_\lambda x\right) \tag{16.1}$$

Equation 16.1 is presented in the context of one dimensional radiative heat transfer. Here, I is the intensity of radiation in the direction along the x-coordinate and suffix λ indicates a dependence on wavelength. Suffix 0 refers to the value of intensity at $x = 0$, namely the initial intensity before radiation enters the region filled with gas. The parameter κ_λ is called *absorption coefficient*; it is dependent on the chemical composition of the gas being studied as well as wavelength. Equation 16.1 is called *Beer's law.* Equation 16.1 shows that the gas has a transmittivity equal to

$$\tau_\lambda = \exp\left(-\kappa_\lambda x\right) \tag{16.2}$$

Hence, it has an absorptivity

$$\alpha_\lambda = 1 - \exp\left(-\kappa_\lambda x\right) \tag{16.3}$$

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[1] unless the particle is transparent or unsymmetric in shape. In the former, considerable anisotropy is generated by refraction, internal and external, in radiation leaving the particle surface.

Chapter 17

Combined Conduction and Radiation

In several applications, heat transfer occurs by the mechanism of conduction as well as radiation. Heat transfer by joint mechanisms of conduction, convection, and radiation is also possible, but is not within the scope of the present book. The features seen in mathematical models of combined heat transfer are revealed in the following examples.

Example 17.1 Heat transfer from a radiating fin.

Consider steady heat transfer from a rectangular fin, whose base temperature is T_b, Figure 17.1. The cross-sectional area is A, perimeter P, and the length is denoted as L. A segment of the fin of thickness Δx at a distance x from the base is at a temperature $T(x)$. The ambient temperature is specified as T_a. The *fin approximation*, namely uniformity of temperature across the fin cross-section is taken to be applicable. The relevant physical mechanisms of interest are heat conduction within the fin and convection+radiation to the environment[1]. The goal of the analysis is to predict the temperature distribution in the fin along its length.

For an infinitesimal segment of length Δx located at point 1 (Figure 17.1), the net diffusive heat flux, namely the difference of heat fluxes crossing its left and the right faces is

$$\text{net diffusive flux} = q(x + \Delta x) - q(x) = -k\frac{d^2T}{dx^2}\Delta x$$

Diffusive heat transfer through the fin of cross-section A is $-kA(d^2T/dx^2)\Delta x$. The difference is accounted for as heat losses to the ambient by convection and radiation. The former is $h_c P\Delta x(T(x) - T_a)$, with h_c representing the convective heat transfer

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[1] In the present chapter, convection is given a simplified treatment *via* a heat transfer coefficient, which is in turn a prescribed quantity. Explicit determination of the heat transfer coefficient is not under discussion.

Chapter 18

Radiative Transfer Equation

Radiative heat transfer between surfaces separated by vacuum or by a transparent medium has been discussed in the previous chapters. In many engineering applications, the enclosure is filled with an absorbing gas, a semi-transperant solid or liquid, or a medium with absorbing and scattering particles including gas bubbles in a liquid phase. Nuclear reactors, plasma and fusion reactors, crystal growth applications involve such *participating* media where radiation intensity and possibly its spectral distribution are altered during the passage of energy from one surface to the other. Radiative energy entering such a domain may be augmented or attenuated due to absorbtion, emission and scattering within the material contained in the enclosure. The present chapter formulates the equation that describes changes in radiation intensity during its passage in a participating medium [1].

18.1 Introduction

The radiation intensity in a given direction is constant along its path while traveling through a transparent medium. This property of intensity makes it a suitable quantity for description of absorption, emission, and scattering of energy within a radiatively participating medium. Any change in intensity along a given path must now be attributed to one or more of these phenomena. The radiation intensity field within a participating medium is denoted as $I_\lambda(\mathbf{r}, \hat{\Omega})$. It is a function of location (position vector $\mathbf{r}$), direction of travel of the beam of radiation (direction vector $\hat{\Omega}$) and wavelength (λ).

At the particulate level, absorption and emission are surface phenomena, analogous to those discussed in the preceding chapters. Scattering, on the other hand, exhibits greater complications both in terms of scale and physical origin. The scattering mechanism depends on the ratio of the particulate diameter (d_p) and wavelength. For $d_p/\lambda >> 1$, the particle casts a shadow in the beam direction. For $d_p/\lambda \approx 1$, radiation

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[1] Material which contributes to augmentation and attenuation of intensity is denoted as participating medium

Chapter 19

Exact Solutions of the Radiative Transport Equation

19.1 Introduction

The conservation of energy principle, in general, leads to a non-linear partial differential equation[1]. The radiative flux in the energy equation requires the calculation of irradiation, an integral that is evaluated in terms of the radiation intensity over the applicable range of solid angles. The solution of the overall energy equation yields the temperature field but requires the estimation of intensity field in the domain of interest. The equation of radiative transport (RTE) derived in the previous chapter is an integro-differential equation for radiative intensity. Thus, for domains involving participating media, the contribution of thermal radiation to the energy balance results in a non-linear integro-differential equation.

For low temperature applications, the contribution of thermal radiation is negligible with respect to the overall heat transfer. More generally, however, radiative intensity is a function of temperature. Hence, the equation of radiative transport (RTE) and the overall energy equation are coupled via the temperature dependent intensity field. Decoupling of these two equations are possible for two simplified cases: (a) known temperature field in the domain, and (b) for radiative equilibrium conditions when thermal radiation is the only mode of heat transfer. In the absence of other modes of heat transfer except radiation, the divergence of radiative heat flux is zero. This results in an integral equation relating temperature with the intensity field via irradiation.

The steady state equation of radiative transfer (RTE) involves six independent variables: the wavelength of radiation, three space coordinates and two coordinates describing the direction of travel of photons. In addition, absorption, scattering and emissive properties of the medium are required to be known for a complete solution. Radiative properties are strongly dependent on wavelength and are usually difficult to measure.

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[1] Nonlinearities may be related to dependence of material properties on temperature, nonlinear boundary conditions, and dependence of the velocity field on temperature.

Chapter 20

Numerical Solution of the Radiative Transport Equation

20.1 Introduction

The mathematical treatment of heat transfer through radiatively participating media is complex. This is because a separate equation of transport (RTE) is required to be solved to determine the contribution of the radiative heat flux to the combined energy equation[1]. The radiative transport equation is of the integro-differential type involving two independent direction coordinates and three spatial coordinates. For a non-gray medium, wavelength will appear as an additional independent variable. In addition, the equation of radiative transport is coupled to the combined energy balance equation. Decoupling is possible for two simplified cases. One, the temperature field is known and so the energy equation need not be solved. The solution of radiative transport equation provides the intensity field. The radiative heat flux can then be obtained from the intensity field. Secondly, if radiative equilibrium prevails, the temperature field can be obtained from the intensity field.

Strictly, neither of these approximations is realized in engineering problems. Hence, an iterative procedure needs to be adopted. The intensity field is first obtained by solving RTE using an initially guessed temperature field. The divergence of radiative heat flux is then estimated from the intensity field. The overall energy balance equation involving the divergence of radiative heat flux is then solved to improve the estimate of the temperature field. Iterations continue till the temperature field attains convergence.

Along with coupling between RTE and the combined energy equation, it is seen that RTE is an integro-differential equation that can seldom be analytically solved. The present chapter deals with the approximate solution of RTE by the process of discretization. The temperature field is taken to be known, at least as an approximation, that can subsequently be improved by iteration.

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[1] The combined equation includes all modes of heat transfer - conduction, convection, and radiation.

Chapter 21

Combined Conduction, Convection and Radiation

21.1 Introduction

To estimate the temperature field in a radiatively non-participating medium, the energy equation that includes the diffusive and the convective terms is solved. The contribution of thermal radiation enters the calculation in the form of a nonlinear boundary condition at the enclosure surfaces. When the surface temperatures are known, the conductive, convective, and radiative, heat fluxes at the enclosure walls can be obtained individually and simply added to obtain the total heat flux[1]. This approach was discussed in Chapter 17. If the medium is a stationary solid exposed to the ambient, heat transfer will occur by conduction with a combined convective-radiative boundary condition for heat transfer to the environment. If the medium is a non-participating flowing fluid, heat transfer will be by conduction and convection with a convection-radiation boundary condition for representing heat transfer to the ambient.

In many instances, the convective field may depend on temperature distribution itself, as can happen when fluid motion is by the action of buoyancy. Here, radiation can alter temperature gradients and hence the strength of convection itself. The additive approach to estimating heat fluxes referred above will not be appropriate. Here, it would be necessary to solve the system of equations of flow and heat transfer with radiative boundary conditions all together, and estimate wall fluxes from the resulting temperature distribution.

For a radiatively participating medium, the radiative heat flux originates from the surface and the medium contributes to its attenuation by absorption and scattering. The flux is augmented by local emissions from the medium itself. These mechanisms lead to the incorporation of the divergence of radiatve heat flux into the overall energy balance for an elementary volume in the domain. The divergence of radiative heat flux involves

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[1] If the material velocity is zero at a wall, the convective flux is also zero. On the other hand, convection steepens temperature gradients at the wall and enhances heat transfer.